

# What can we know about that which we cannot even imagine?

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## Abstract

In this essay I will consider a sequence of questions, ending with one about the breadth and depth of the epistemic limitations of our science and mathematics. I will then suggest a possible way to circumvent such limitations. I begin by considering questions about the biological function of intelligence. This will lead into questions concerning human language, perhaps the most important cognitive prosthesis we have ever developed. While it is traditional to rhapsodize about the perceptual power provided by human language, I will emphasize how horribly limited – and therefore limiting – it is. This will lead to questions of whether human mathematics, being so deeply grounded in our language, is also deeply limited. I will then combine all of this into a partial, sort-of, sideways answer to the guiding question of this essay: what we can ever discern about all that we cannot even conceive of?

I'd like to consider a sequence of questions in this essay, each one leading into the next, somewhat like successive chapters in a monograph. The first of these questions concern reasons for believing our cognitive capabilities are extraordinarily limited, why the vaunted power of our mathematics and science are actually manifestations of its extremely constrained nature. This leads into questions concerning whether our cognitive abilities might forever be limited this way, and the nature of the consequent limitations on what we might ever be able to glean concerning what it is that we will never even be able to imagine. The first of these questions have been raised before, but the ending ones are novel, I believe, at least in their emphasis.

These questions concern the possibility of cognitive constructs beyond the capabilities of human minds. Unfortunately, though, I am a mere human (as is the reader, I presume). So my language in formulating and discussing questions

on this topic will necessarily be allusive and somewhat imprecise. Nonetheless, I hope the essence of the argument is clear. To that end, I finish this essay with some suggestions for how to answer the questions I raise empirically, thereby (I hope!) rescuing this essay from the vast swamps of semi-philosophical BS to be found in the libraries of humankind.

To begin this journey of questions, note that high levels of abstract intelligence, high ability to reason about issues that do not only involve control of our physical bodies, seems to have been exceedingly rare across all Terran species. This is true even if we consider the entire history of life on Earth, rather than restrict attention to the set of species currently inhabiting our biosphere.

Indeed, for vast stretches of time the highest level of intelligence on Earth seems to have increased very slowly, at best. Even if there were not an immediate adaptive fitness benefit to a species for increasing its intelligence level, one might expect that the random exploration inherent in biological evolution would have resulted in a stochastic “drift” increasing the maximal level of intelligence over all species in the biosphere, much like the drift process that played a part in the increase in the complexity of the biosphere as a whole [69].

One way to investigate this phenomenon is to adopt a functionalist, teleological perspective on evolution, viewing it as a process that creates increasingly “optimized” designs of biological organisms. It may simply be that it may take a billion years to design a brain with our levels of intelligence, if the design is done using the kinds of “search algorithm” embodied in natural selection. Human brains are physically complicated devices. Presumably much of that complexity is necessary in any system that has the cognitive capabilities of our brains. And presumably, such complexity requires a while to evolve under the “search algorithm” embodied in natural selection.

While it’s hard to address this possible explanation quantitatively, it seems implausible. No other significant capability of biological organisms — and there are some capabilities that are implemented with very complicated designs — took a billion years of tinkering to evolve. Other biological capabilities, ranging from ability to navigate through an environment based on patterns of photons reflected from that environment, to the ability to defend against a vast range of microbial assaults, to the ability to disentangle multiple simultaneous sets of coupled waves of atmospheric disturbances and thereby solve “the cocktail party problem”, all of these abilities involve very complicated mechanisms. And yet all are extremely common across Terran species, both at present and in the past.

Other possible explanations for the rarity of high levels of abstract intelligence have other such problems — except for the simplest explanation, the one suggested by Occam’s razor. The rarity of high levels of abstract intelligence among Terran species strongly suggests that the fitness costs associated with such intelligence are high compared to the typical associated fitness benefits. It may simply

be that evolutionarily speaking, it is stupid to be smart.

Indeed, it seems that at least the sensory-motor information processing in our brains involve all kinds of algorithmic shenanigans so that we can actually use our brains as little as possible, so that we can think as little as possible. Phrased in terms of a sloppy and hackneyed metaphor, it seems that the software running on our brains is designed so that it involves as few executed lines of code as possible. (See for example the vast literature on the brain's use of predictive coding [8, 92] and sparse coding [80, 91], as well the literature on the low-level processing in the visual system [86, 24, 96] in addition to coding of more abstract concepts in the brain [9, 47].)

It is often argued that the underlying reason for this aversion to thinking is to reduce the associated fitness costs [13]. Indeed, such costs to thinking are not difficult to find. In particular, it turns out that brains are extraordinarily expensive metabolically on a per-unit-mass basis, far more than almost all other organs (the heart and liver being the sole exceptions — see [27, 93, 70, 14]). Consistent with this, it is not just that the software comprising our minds that seems tailored to reduce metabolic costs; the hardware supporting that software — the physical architecture of our brains — also seems tailored to reduce metabolic costs.

We do not have a good understanding of exactly *how* our hardware is used to provide the ability of humans to engage in activities requiring high levels of abstract intelligence. We do not understand how “brain makes mind” [18]. Indeed, the brain seems to incur major energetic costs even when it is in the resting state, not currently engaged in any “task-focused” activities, but simply providing the ability to engage in such activities [84]. Nonetheless, it seems clear that less metabolic cost  $\Rightarrow$  less brain mass  $\Rightarrow$  less abstract intelligence.

In light of this advantage of low intelligence, one would expect humans to have the minimal possible computational abilities sufficient for surviving in the particular ecological niche in which we were formed (namely, the niche of social omnivores that inhabit African mixed forest / savanna). One would expect our computational abilities to be extremely limited. Indeed, make the simple assumption that there are *some* cognitive abilities that are not directly helpful for surviving in our ecological niche (and that not inevitable side-effects of any such abilities that are helpful), but yet are crucial to formulating a science and mathematics (hereafter abbreviated as SAM) that fully encompasses the nature of physical reality. Then due to the evolutionary costs of cognitive abilities, we would have to predict that we do not have such abilities that are not directly helpful for our survival.

Biology also provides some purely statistical reasons, not involving arguments based on Darwinian natural selection, which also suggest that our cognitive abilities are deficient. As a first example, assign a bit to every *other* species on Earth not us humans, saying whether that species (by our lights) has deficiencies in its

cognitive abilities, whether there is at least one kind of reasoning / inference that we are sure that the species is incapable of. There is only one species we cannot assign such a bit to: ourselves, since we do not know whether we have such deficiencies. But we can assign it to all the other billions of species on Earth. Moreover, every single one of those billions of other species has the same value for the bit: yes, they are deficient, cognitively. If we applied an admittedly crude Bayesian argument to this particular dataset, along the lines of Laplace’s law of succession, we would conclude that it is very likely that the one species for which we have no data — ourselves — has such deficiencies as well.

As a second example of such an argument, fix some way of formalizing a “type” of cognitive capability. The ability to sense the external world would be one such “type” of cognitive capability; the ability to remember past events would be another; the ability to plan a future sequence of actions would be another, etc. Given any such formalization, we could consider the union of all kinds of cognitive capabilities possessed by all organisms on Earth at any given time. We can then conduct a time-series analysis of how the size of that union of all kinds of cognitive capabilities of all species on Earth has changed over evolutionary history.

Arguably, no matter what precise time-series analysis technique we use, and no matter how we formalize “type of cognitive capability,” we will conclude that the trend line of that union’s size currently has a strictly positive slope. After all, in no period has the set of all types of cognitive capabilities held by any entity in the terrestrial biosphere *shrunk*; the biosphere as a whole has never lost the ability to engage in certain kinds of cognitive capability. Accordingly, we will conclude that there is not just growth over time in the *degree* of cognitive capability of terrestrial species, but a growth in the set of *kinds* of cognitive capability. If we simply extrapolate that trend into the future, we’re forced to conclude that there are likely to be kinds of cognitive capabilities that some future organisms will have but that no currently living Terran species has — including us.

Given this, it should not be surprising that the list of our computational limitations is legion. To give one striking example, evidently it was not even sufficiently beneficial in terms of adaptive fitness for us to have working memory capable of holding more than seven enumerated objects at once. Indeed, modern digital computers, which are exceedingly simple systems in comparison to the complexity of the human brain, vastly outperform us computationally in myriad ways [26, 44, 89]. Moreover, the small set of those cognitive tasks that we can still perform better than our primitive digital computers is substantially shrinking from year to year.<sup>1</sup>

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<sup>1</sup>It is important to appreciate that the rate of this shrinkage is significant, notwithstanding the fact that there are, for now, some important cognitive tasks that elude AI programs [77], and

Despite all our preening in front of our collective mirror about how smart we are, it seems that we must in fact be very dumb organisms, by any objective criteria, in terms of abstract intelligence. Thanks to the exigencies of natural selection, we must have highly restricted cognitive abilities — the barest minimum needed to scrape through a few million years of hunting and gathering until we got lucky and stumbled into the Neolithic revolution. It may simply be that that the presumption stated at the beginning of this essay, that we “have high level of abstract intelligence” is, objectively speaking, wrong. It may simply reflect camouflaged biases we have on the topic of “intelligence”, esteeming the type that we happen to have above all others.

However, before getting too comfortable in this conclusion reached by considering the time series of intelligence on Earth, there is a peculiarity of the time series that we must explain. It seems that the maximal intelligence on Earth was gradually, smoothly increasing until around 50,000 years ago. At that time there began a major jump, when modern *homo sapiens* started on the trajectory that would ultimately produce all of modern science, art, and philosophy. Is this the case? Are we in fact in the midst of a major jump in the cognitive ability of organisms on Earth?

This provides the first question of this essay:

1. Why is there a major chasm with the minimal cognitive capabilities necessary for survival by pre-Holocene hominids on one side, and on the other side, all those cognitive capabilities that Kurt Gödel, Albert Einstein, and Ludwig van Beethoven called upon when conjuring their wonders?

Why is there the major chasm between the computational skill set needed to avoid the proverbial lion lurking in the bush, and the computational skill set that Lady Murasaki, Vyasa, and Al Khwarazmi exploited when they wished upon their separate stars? As the canard goes, there is no evident fitness benefit for a savanna-forged hairless ape to be able to extract from the deepest layers of physical reality cognitive palaces like the Standard Model of particle physics, Chaitin’s incompleteness theorem, or the Zen parable, “Ten Verses on Oxherding.” And in fact, by the arguments above, there are likely major fitness costs to our having such abilities. So why do we have them?

We can gain some perspective on this question by reformulating it in the context of the other species in the terrestrial biosphere:

2. Restricting attention to what are, in some sense, the most universal of humanity’s achievements, the most graphic demonstrations of our cognitive abilities:

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notwithstanding the research frustrations endured by some the very earliest of AI researchers, sometimes summarized as “Moravec’s paradox” [25, 76, 78].

*Why were we able to construct present-day science and mathematics, but no other species ever did? Why are we uniquely able to decipher some features of the Cosmic Baker's hands by scrutinizing the breadcrumbs that They scattered about the universe? Why do we have that cognitive ability despite its fitness costs? Was it some subtle requirement of the ecological niche in which we were formed — a niche that at first glance appears rather pedestrian, and certainly does not overtly select for the ability to construct something like quantum chromodynamics? Or is our ability a spandrel, to use Gould and Lewontin's famous phrase — an evolutionary byproduct of some other trait? Or is it just a cosmic fluke?*

This is hardly a new question, of course. But it gives rise to a sequence of many other, less commonly considered questions. To begin:

3. Are we really sure that no other species ever constructed some equivalent of present-day SAM? Are we really sure that no other apes — or cetaceans or cephalopods — have achieved some equivalent of our SAM, but an equivalent that we are too limited to perceive?

I would say that this question at least we can answer, and the answer is “yes.” It may well be that, if and when we come to partially decipher the “communication” of some non-human species [10], we will find messages in their communications that in some sense correspond to our SAM. For argument's sake, let's suppose that this *will* be the case — that we will find such science(ish) and math(ish) messages in the communications among members of some other species. Even so, we would still have to conclude that their SAM is inferior to ours in some extremely important respects. After all, incontrovertibly, the Earth's physical environment reflects the effects of our understanding of SAM — it reflects our activities — far more than it reflects the activities of any other single species [15].

There is no reason to believe that other species would not have also completely transformed the biosphere to serve their ends, just as we have, if only they had the SAM that is necessary to do so. The implication, given the absence of such a transformation by some other species, is that no such species has developed as sophisticated a SAM as we have.

There is nothing subtle about this evidence of our singular abilities in SAM. Nor is it particularly biased to invoke this evidence to distinguish us from other species. After all, the evidence of our abilities in SAM, given by our transformation of the biosphere, is obvious to members of other species (many of whom we've actually wiped out), whereas there is no evidence of *their* abilities in SAM that is similarly obvious to us.

This answer to question (3) suggests that we should modify question (2):

4. If the evidence of the uniqueness of our SAM is the modifications that we, uniquely, have wreaked upon the terrestrial biosphere, should the question really be why we are the only species who had the cognitive abilities to construct our SAM *and* were able to build upon that understanding, to so massively re-engineer our environment? To give a simple example, might some cetaceans even exceed our SAM, but just do not have the physical bodies that would allow them to exploit that understanding to re-engineer the biosphere in any way? Should the focus of the inquiry not be whether we are the only ones who had the cognitive abilities to construct our current SAM, but rather should the focus be expanded, to whether we are the only ones who had both those abilities *and the ancillary physical abilities* (e.g., opposable thumbs) that allowed us to produce physical evidence of our SAM?

Note that the combination of our SAM and our physical ability to exploit that SAM has not only provided us with the ability to twist the biosphere to satisfy our desires, but also with cognitive prostheses [52] and extended minds [35, 37, 34]. Furthermore, the capabilities of those extended minds have been greatly magnified — by the cumulative collective process of culture and technological development [43, 46, 62]. In turn, these extended minds have accelerated the development of culture and technology. There is a feedback loop between them.

This feedback loop has allowed us to “build out” the set of our original cognitive capabilities, to expand its breadth in an exponentially growing dynamics that is not solely driven by genotypic evolution — which is how the expansion of our cognitive capabilities was driven before the Pleistocene and is still driven for all other Terran species. Concretely, this combination of high cognitive abilities and the physical ability to exploit them may not only have allowed us to expand our SAM beyond whatever other Terran species have already achieved, but even beyond what other Terran ever *can* achieve. Indeed, it may have been this feedback loop which allowed the gap to form between our cognitive capabilities and the minimum such abilities necessary for survival on the Veldt, i.e., it may provide the answer to question (1).

However, a perhaps crucial feature of the feedback loop is that while it has inflated our *original* cognitive capabilities, it is not clear that it has provided us any *wholly new* cognitive capabilities. In fact, it might not ever be able to. The future phenotypic possibilities of any species evolving according to genotypic evolution are constrained by the frozen accidents in its evolutionary past [5, 38, 72], which limit its possible future forms. This may not only be true for relatively slow genotypic evolution; it may be just as true if the evolution is very fast and driven by a feedback loop between extended minds and collective culture. Perhaps our future SAM will be constrained by the set of cognitive capabilities we had when we

started to construct our current SAM.

This suggests a different kind of resolution to question (1). Maybe the gap between our current SAM, on the one hand, and the kinds of knowledge our minds were designed to uncover by traipsing across the infamous Veldt, on the other hand, perhaps that gap is not a “chasm.” Perhaps it is more accurate to describe the gap as a small divot, with the full range of all that will forever be unknown to us, all that our SAM can never encompass, extending far beyond our position just on this side of the gap.

Yes, sure, perhaps our SAM and our ability to exploit our SAM far exceeds that of other species on Earth. But perhaps these are tiny differences, almost invisible on the scale of what would accompany cognitive capabilities that are more extensive than those that we currently have or will ever have. Maybe the answer to Wigner’s famous question, about why our mathematics is “unreasonably effective” at capturing the nature of our physical reality [97], is that our mathematics *isn’t very effective at all*, that in fact our mathematics can only capture a tiny sliver of reality. Perhaps the only reason it *appears* to be so effective is because our vista is so restricted, to just those very, very few aspects of reality that we can conceive of.<sup>2</sup>

Therefore, perhaps the interesting question is not why our augmented minds seem to have abilities greater than those necessary for the survival of our ancestors, but rather whether our augmented minds have the minimal abilities necessary for grasping reality:

5. Ancillary abilities or no, are we unavoidably limited to enlarging and enriching the SAM that was produced by our species with the few cognitive abilities we were born with? Is it impossible for us to concoct wholly new types of cognitive abilities — computational powers that are wholly novel *in kind* — which in turn could provide us wholly new kinds of SAM, kinds of SAM that would concern aspects of physical reality currently beyond our ken?

Perhaps the earliest versions of this question in modern, Western literature were throwaway phrases scattered among various essays. In these early, verbal baubles it was suggested that the universe may be “queerer / stranger / odder than we can suppose / imagine / conceive” [57, 36]. Having other fish to fry, the authors of these early texts rarely fleshed out what they meant with these phrases of theirs. In fact, they often seemed to mean that the universe may be stranger than we can *currently* imagine, due to limitations in current scientific understanding, rather than being stranger than we can ever imagine, with any future science, due to inherent limitations of what we can ever do with (future efflorescences of) our

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<sup>2</sup>See [59] for similar sentiments concerning Wigner’s question.



minds. (For example, [36] explicitly considers the first type of strangeness when he introduces the phrase, while [57] explicitly predicts that we will “one day ... be able to look at existence from the point of view of non-human minds”, which seems to rule out the second type of strangeness.)

Various forms of question (5) were then raised later with a bit more care and detail [16, 31, 51, 68, 82]. However, most of those later forms of the question have concerned the dubious “hard problem of consciousness” [61] and the closely related “mind — body problem” [75].

Fortunately, we can approach question (5) in a more rigorous manner. To see how, we can start with the recently (re)popularized idea that our physical universe in general, and so in particular we denizens of that physical universe, might just be part of a simulation produced in a computer of some super-sophisticated race of aliens [21, 58, 30].<sup>3</sup> This idea can be extended in an obvious manner, simply by noting that the aliens that simulate our universe might themselves be a simulation in the computer of some even more sophisticated species, and so on and so on, in a sequence of ever-more sophisticated aliens. Going the other direction, in the not too distant future we might produce our own simulation of a universe, complete with entities that have “cognitive capabilities”. Indeed, we might produce such a simulation whose cognitive entities can produce their own simulated universe in turn, etc. So there might be a sequence of species’, each one with a computer running a simulation that produces the one just below it in the sequence, with us somewhere in the middle of that sequence.

This question of whether we are a simulation or not is answered rather trivially if we adopt ontic structural realism, especially if it is formalized in terms of Tegmark’s level IV multiverse: yes, in some universes we are a simulation, and no, in some other universes we are not. (See also [28].) For argument’s sake though, let’s restrict attention to universes in which we are indeed a simulation. This would raise an obvious question:

6. Is possible for one species, at one level of the sequence of {computers running simulations of computers that are running simulations of ...}, to itself simulate a computer that is higher up in the sequence that it is?

If the answer to this question is ‘no’, that suggests that no matter what we humans, in our physical universe, can do with our brains, it is a strictly smaller subset of what can be done by the aliens and their computers who reside higher up than us in the sequence of simulations.

Of course, the answer to question (6) depends on the precise definitions of terms like “simulate” and “computer”. As it turns out, the theories of formal sys-

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<sup>3</sup>Note that this idea presupposes a form of the Church-Turing thesis, or at least ontic structural realism [54, 74, 73, 7] and related concepts grounded in modern physics, like Tegmark’s level IV multiverse [94, 95].

tems and computer science provide many theorems which suggest that the answer to question (6) is indeed ‘no’, whatever such precise definitions we adopt.

To give an example of such theorems, pick any formal system powerful enough to be subject to Godel’s incompleteness theorems. These theorems establish that certain propositions statable in that formal system cannot be proven in that formal system. Broadly speaking though, all one needs to do to nullify such a conclusion is to introduce an extra axiom into the formal system that states whether the associated proposition is true or not. Bang — the apparently paradoxical statement giving the impossibility theorem has been resolved.

Unfortunately though, the combination of the original formal system with that new axiom is, strictly speaking, a more powerful formal system than the original one. Moreover, *that* new, more powerful formal system is itself subject to exactly the same kind of Godelian impossibility theorems as those applying to the original formal system, just referring to a different proposition. And of course, those new impossibility theorems in that new formal system could themselves be resolved by introducing yet another axiom, resulting in a yet more powerful formal system [17]. This sequence of increasingly powerful systems has no upper limit — each new formal system is strictly more powerful than the ones below it whose incompleteness theorems it obviates.

These fundamental properties of formal systems provide one way to make the scenario considered by Bostrom et al. more precise. Depending on how precisely one wants to define the word “simulate”, one can identify each member of the sequence of increasingly powerful formal systems with one of the “computer simulations produced by sophisticated aliens” in the nested-dolls sequence of simulations considered by Bostrom, et al. Crucially, no matter how we augment a proof in a formal system, no matter how we combine it with other proofs in that formal system, there will be proofs in the formal systems higher in the hierarchy that it can never replicate.<sup>4</sup> In this sense, *no matter how our physical brains are organized, and no matter what physical systems we ever construct to augment our brains*, we will be cognitively weaker than the aliens residing in those other, more powerful formal systems, peering down at us (assuming that the cognitive abilities of those aliens includes deciding all decidable propositions in the formal system that defines us).

Another formalization of the musings of Bostrom et al. involves various “reduction hierarchies” (arithmetic and analytic hierarchies, and in their resource-

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<sup>4</sup>Interestingly, while there is no upper limit to this sequence of formal systems / simulations, there *is* a lower limit; it is the formal system that arises in Godel’s incompleteness theorem, whose axioms include those of Peano’s arithmetic. (In weaker formal systems it is not even possible to perform Peano arithmetic operations.) In this sense, there is an infinite set of “simulations” that are strictly more computationally powerful than the one giving our physical universe — but none that we would consider computationally interesting that are weaker.

limited forms, the polynomial and exponential hierarchies). The computer science form of these hierarchies involve decision problems which we wish a Turing machine to solve.<sup>5</sup> Very loosely speaking, define an “oracle” as a system that can instantly decide whether a given decision problem is in some special set  $\alpha$ . Choose a set  $A \subset \alpha$  of the decision problems in  $\alpha$  that can be decided by a Turing machine subject to a certain time complexity (e.g., polynomial complexity P or non-deterministic complexity NP). Informally, this means that all of the decision problems in  $A$  are “easy” to solve, as far as that Turing machine is concerned, while problems lying outside of  $A$  are so difficult as to be effectively impossible to solve. No matter how we program that Turing machine, no matter how we “augment” one program running on it with another one, it will always find problems outside of  $A$  effectively impossible.

Next, define a larger set of problems,  $B$ , which contains all the problems in  $A$ , and which can be decided with the same time complexity as those in  $A$  (i.e., that are “just as difficult to decide” as those in  $A$ ), by now allowing that same Turing machine to call an “oracle” that can instantly decide whether a given problem is in  $\alpha$  or not, and do so an arbitrary number of times. This new, souped-up Turing machine, with access to the oracle, is strictly stronger than the original Turing machine, in the simple sense that there is a larger set of decision problems, containing the original set, that are all easy for the new Turing machine. (In contrast, problems in  $B$  that are outside of  $A$  are strictly harder to decide for the original Turing machine, which did not have access to the oracle.)

We can iterate this “oracle-izing” procedure, by defining a new oracle that can instantly decide whether a given problem is in  $B$ . (Note that  $\alpha \subseteq A \subseteq B$ .) By allowing our original Turing machine to call this new oracle an arbitrary number of times, we allow it to decide a set of problems,  $C$ , which is even larger than  $B$ , again with the same time complexity as the original Turing machine decided the original set of problems  $A$ . In this way we define an entire infinite “hierarchy” of increasingly powerful Turing machines, able to decide an increasingly bigger set of decision problems without expending extra effort. Crucially, no matter how we write the programs of the Turing machine at one level in the hierarchy, no matter how we “augment” it with other programs run on the same machine, there will always be problems that are easy for the Turing machines at higher levels in the hierarchy to solve, but are effectively impossible for any Turing machine at its level.

This argument involving hierarchies of increasingly powerful Turing machines, able to decide ever larger sets of decision problems with the same amount of ef-

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<sup>5</sup>More formally, they involve bit strings codifying decision problems that must be assigned either the value ‘true’ or the value ‘false’ by running an appropriately programmed Turing machine that takes that bit string as input. See [11].

fort, is analogous to the argument above concerning a sequence of increasingly powerful formal systems, each of which resolves the impossibility theorems of the formal systems below it in the sequence but is weaker than the formal systems above it. The implication is analogous as well: no matter how we augment our minds — no matter where we lie in the infinite sequence of ever more powerful oracles — we will be more limited in our cognitive abilities than the minds higher in the sequence.<sup>6</sup> In summary, depending on how exactly one wants to define the word “simulate”, the concerns of Bostrom, et al., properly formalized, strongly suggest that augmenting our brains can never allow us to fully grasp / cognize / perceive our physical reality.

These are some examples of the *content* of our mathematics that suggest that we are too limited in our cognitive abilities to cognitively engage with reality in its full extent. There are other aspects of our mathematics that instead involve its *form*, not its contents, which also strongly suggest that we are highly constrained in our cognitive capabilities.

One of the most powerful fields of mathematics to develop in the past half century is category theory [12, 85]. Category theory provides an extraordinarily concise way of unifying almost all fields of (human) mathematics, ranging from set theory to algebra to topology to analysis. By stripping out the extraneous details, it makes glaringly obvious that all our fields of mathematics are just variants of the same underlying structures, known as “categories”. What makes category theory so beautiful is that the definition of a category is almost absurdly simple: a category consists of abstract “objects”, together with ordered pairs of objects, which are called “morphisms”. There are a few simple requirements imposed on how the morphisms in a category must be related to one another. *And that is it.* All of our mathematics reduces to embellishments of these simple structures.

One response to category theory — part of the reason it strikes so many people as beautiful — is to be stunned by it. Category theory shows us that the many fields of human mathematics, which we are used to viewing as many different types of flower filling a great garden, are in fact many instances of the same underlying, singular plant, planted all over our yard, simply viewed from many different angles. This response might lead one to suppose that the simplicity of category theory reflects some kind of simplicity of mathematical reality.

However, being a bit more objective about things, it might just as well be that the simplicity of the mathematical garden we have tilled reflects a limit of *us*, the gardeners, not of the number of species of mathematical plant there are. It may be that the reason that our fields of mathematics are all embellishments of

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<sup>6</sup>More precisely, no matter how we program a Turing machine to exploit whatever oracles our minds have access to, there will be other Turing machines, able to access other oracles, that once they are properly programmed can solve a larger set of decision problems than we can for the same computational effort.

a simple underlying structure is that they were created by simple-minded beings: us. It might be that there is a vast expanse of mathematical reality which *cannot* be expressed in such an extremely simple form, but is intrinsically more complex — and is forever beyond us.

This leads to the following question:

7. Is the very *form* of the SAM that we humans have created severely constrained? So constrained as to suggest that the cognitive abilities of us humans — those who created that SAM — is also severely constrained?

One does not have to learn category theory to address this question. If one steps back from our SAM, viewing it from afar, one notices that there are striking limitations in how our SAM is formulated, limitations that are common to the SAM produced by almost all human societies. These are limitations in the kinds of patterns with which our SAM is *represented*. The near universality of these limitations strongly suggests that we humans are incapable of constructing any SAM unless it can be represented using patterns that obey those limitations — that the limitations are an unavoidable inheritance of the Veldt-designed cognitive capabilities of our distant ancestors. These limitations in how SAM is represented have long been explicitly recognized, and in fact are central to how modern mathematicians and philosophers conceive of mathematics — it's just that nobody has emphasized that they are *limitations*.

As was explicitly recognized at least as far back as Wittgenstein [19, 87, 98], all of human mathematics consists of discrete sequences of clearly delineated patterns composed either of marks on a surface or words in speech. All statements in mathematics consist of finite sequences of elements from a finite set; e.g., “ $1 + 1 = 2$ ” is a sequence of five elements from a finite set of symbols. All proofs in mathematics — all theorems based on Zermelo — Fraenkel Choice (ZFC) set theory, all predicate logics, all category theory — comprise a finite discrete sequence of such statements.

Moreover, to paraphrase Galileo, all of our current physics, and so all of our formal understanding of the foundations of physical reality, is written in the language of our mathematics [66, 94, 95, 108]. Even if we consider the less formal sciences, they are still formulated in terms of human language, in addition to mathematics — and therefore in terms of finite strings of symbols, since like mathematics, human language comprises finite strings of symbols. Our evident restriction to manipulating such strings is a major limitation on how we represent both mathematical and physical reality.

Much of philosophy has reacted to this observation, that our SAM is just a set of finite sequences of symbols, by trying to unpack / formalize the precise

way that such finite sequences can “refer to something outside of themselves.”<sup>7</sup> The field of mathematics has reacted in a similar way, expanding formal logic to include modern model theory [63] and metamathematics [67].

Arguably though, what is truly stunning about the fact that modern SAM is formulated in terms of finite sequences is its exclusivity; *nothing* other than such finite chains of symbols is *ever* found anywhere in modern mathematical reasoning. So fundamental is this restriction in current mathematical reasoning that it has implicitly been enshrined in our current definition of such reasoning via the Church — Turing thesis<sup>8</sup>.

This gives rise to our next question:

8. Is this restriction to finite sequences somehow a necessary feature of any complete formulation of physical reality? Or does it instead reflect a limitation of how we humans can formalize any aspect of reality, i.e., is it a limitation of our brains?

This question, in turn, immediately gives rise to many others:

9. In standard formulations of mathematics, a mathematical proof is a finite sequence of “well-formed sentences”, each of which is itself a finite string of symbols. All of mathematics is a set of such proofs. How would our perception of reality differ if, rather than just finite sequences of finite symbol strings, the mathematics underlying our conception of reality was expanded to involve infinite sequences, i.e., proofs which do not reach their conclusion in finite time? Phrased concretely, how would our cognitive abilities change if our brains could implement, or at least encompass, super-Turing abilities, sometimes called “hyper-computation” (e.g., as proposed in computers that are on rockets moving arbitrarily close to the speed of light [2])? Going further, as we currently conceive of mathematics, it is possible to embody all of its theorems, even those with infinitely long proofs, in a single countably infinite sequence: the successive digits of Chaitin’s omega [71].

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<sup>7</sup>To appreciate some of the difficulty of this problem, note that any sequence of symbols has no more significance in and of itself about any one specific mathematical or physical object than do the sequences one might find in the entrails of a sacrificed sheep, or in the sequence of cracks in a heated tortoise shell. How to circumvent this issue is called the “symbol-grounding problem” in philosophy [50, 60].

<sup>8</sup>Whether the physical universe obeys the Church–Turing thesis has been subject to endless argument [3, 81], especially in the context of quantum mechanics [79]. In particular, some researchers have designed purely theoretical, contrived physical systems that are uncomputable in some sense or other [83, 39, 88] (see also [29]). My concern here is more fundamental, with whether the implicit assumption of the thesis, that the physical universe can be formulated in terms of a set of sequences of symbols, is valid.

(This is a consequence of the Church — Turing thesis.) How would mathematics differ from our current conception of it if it were actually an uncountably infinite collection of such countably infinite sequences rather than just one, a collection which could not be combined to form a single, countably infinite sequence? Could we ever tell the difference? Could a being with super-Turing capabilities tell the difference, even if the Church — Turing thesis is true, and even if we cannot tell the difference?

Going yet further, what would mathematics be if, rather than countable sequences of finite symbol strings, it involved uncountable sequences of such symbol strings? In other words, what if not all proofs were a discrete sequence of well-formed finite sentences, the successive sentences being indexed by counting integers, but rather some proofs were continuous sequences of sentences, the successive sentences being indexed by real numbers? Drilling further into the structure of proofs, what if some of the “well-formed sentences” occurring in a proof’s sequence of sentences were not a finite sequence of symbols, but rather an uncountably infinite set of symbols?<sup>9</sup> If each sentence in a proof consisted of an uncountably infinite set of symbols, and in addition the sentences in the proof were indexed by a range of real numbers, then (formally speaking) the proof would be a curve — a one-dimensional object — traversing a two-dimensional space. Going even further, what would it mean if somehow the proofs in God’s book [6] were inherently *multidimensional* objects, not reducible to linearly ordered sequences of symbols, embedded in a space of more than two dimensions?

Going further still, as mathematics is currently understood, the sequence of symbol strings in any proof must, with probability 1, obey certain constraints. Proofs are the outcomes of deductive reasoning, and so certain sequences of symbol strings are “forbidden”, i.e., assigned probability 0. However, what if instead the sequences of mathematics were dynamically generated in a stochastic process, and therefore unavoidably random, with *no* sequence assigned probability 0 [108, 33, 45, 53]? Might that, in fact, be how our mathematics has been generated? What would it be like to inhabit a physical universe whose laws could not be represented unless one used such a mathematics [40, 55, 56]? Might that, in fact, be the universe that we do inhabit, but due to limitations in our minds, we cannot even conceive of all that extra stochastic structure, never mind recognize it?

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<sup>9</sup>It should be noted that there *has* been some research on logical systems in which the individual sentences are allowed to involve countably infinite sequences of symbols [1]. Of course, even this “infinitary logic” is formulated in terms of (finite sequences of) finite sequences of symbols, i.e., the equations defining infinitary logic, and the resultant formal investigations of infinitary logic, all occur in papers and textbooks that are finite.

As a final leap, note that all of the suggested extensions of the form of current human mathematics I have just described are themselves presented in terms of ... human mathematics. Embellished with colloquial language, I described those extensions by using the formal concepts of uncountable infinity, multidimensionality, Turing machines, and stochastic processes — all of which are constructions of human mathematics, and so involving finite sets of finite sequences of symbols. What would a mathematics be like whose very form could not be described using a finite sequence of symbols from a finite alphabet?

To gain some insight into these questions, note that the highly limited form of all of human mathematics — sequences of finite strings of symbols — just happens to be *exactly* the structure that we humans use to converse with one another: the structure of human language. Indeed, starting back with Wittgenstein, it has become commonplace to *identify* mathematics as a special case of human language, casting its structure explicitly as grammar in the same sense as grammars arise in human conversation. Note as well that it is a poetic cliché that libraries, and in particular mathematics libraries, are places where we converse (!) with past minds. The implication is that the contents of all the mathematics textbooks in those libraries is part of a conversation; i.e., an exercise in human language. Indeed, historically, mathematics textbooks and papers developed from written correspondence, i.e., from exercises in human language.

So, the form of human mathematics, and of our SAM more generally, just happens to exactly coincide with the form of inter-human communication. Some writers have pointed this out before, that human language’s design matches that of formal logic and Turing machine theory [16]. They have taken this as a wonderful stroke of fortune, that we just so happen have a cognitive prosthesis — human language — that is capable of capturing formal logic. After all — they presume — this means we are capable of capturing all the laws of the physical universe.

A cynic might comment with heavy irony, “Gee, how lucky can you get? Humans have exactly the cognitive capabilities needed to capture all aspects of physical reality, and not a drop more!” This cynic might go on to wonder whether an ant, who is only capable of formulating the “rules of the universe” in terms of pheromone trails, would conclude that it is a great stroke of fortune that they happen to have the cognitive capability of doing precisely that; or whether a phototropic plant would conclude that it is a stroke of fortune that they happen to have the cognitive capability to track the sun, since that must mean that they can formulate the rules of the universe.

Sure, it’s possible that it is just a coincidence, that for some unknown reason the deepest nature of physical reality is expressible in terms of one of our cognitive prostheses. But it certainly seems as plausible that social computation is



simply the most sophisticated cognitive prosthesis we have ever developed, and that even exploiting it to the hilt only allows us to capture a sliver of physical reality. Yes, our science and mathematics — or more precisely, what they seem to be developing into — may be a complete description of *what we understand physical reality to be*. They might be developing into a complete description of what is experimentally accessible to us, even if only indirectly, both now and in the future [94, 95, 108, 109]. But in an exactly parallel manner, a putative ant-level theory of reality in terms of pheromone trails and environmental chemical signals could capture all that ants “understand physical reality to be”, of all that ants can “experimentally access”. And just as there is a huge expanse of physical reality lying beyond the charmed sliver that ants can conceive of, it may be that there is a huge expanse of physical reality beyond our ability to even conceive of.

After all, social computation — human language — was developed for communal sessions of shooting the shit around the campfire after a successful mastodon hunt (plus a few other purposes). There is no reason to believe that features well-suited for such exercises in nocturnal braggadocio can also be used to glean substantial insights into the shape of the hands of the Cosmic Baker, based solely on some crumbs we have discovered, scattered on their kitchen floor.

Noam Chomsky [32], Daniel Dennett [41] and others [42] have marveled at the fact that human language allows recursion, i.e., that it allows arbitrary finite, discrete sequences of symbols from a finite alphabet. They marvel at the fact that humans can create any of a countable number of sentences, viewing this as allowing an amazingly large set of human languages. In contrast, I marvel at how restricted human language is, and so how restricted human SAM is. I marvel at the fact that that limitation appears to be universal. After all, suppose that it were the case that all of physical reality could in fact be formulated in terms of some finite sequence of symbol strings from a finite alphabet. Even so, it would still be striking that, as humans converge to that correct formulation, we have never considered formulations that are not representable this way, perhaps due to considering an erroneous SAM, or perhaps due to choosing an inefficient representation of SAM. I marvel at the small size of the sandbox we play in while formulating our SAM.

This perspective leads to our next question:

10. Is it a lucky coincidence that all of mathematical and physical reality can be formulated in terms of our current cognitive abilities, including, in particular, the most sophisticated cognitive prosthesis we currently possess: human language? Or is it just that, tautologically, we cannot conceive of any aspects of mathematical and physical reality that cannot be formulated in terms of our cognitive capabilities?

To gain some appreciation for this question, note that (presumably) no parame-

cium can even *conceive* of the concept of a “question,” concerning issues that have no direct impact on its behavior, despite how obvious that concept seems to us. Not only would a paramecium not understand the possible answers we have considered for our questions concerning reality, it would not understand the questions, as has often been noted [23, 22]. More fundamentally though, no paramecium can even conceive of the idea of posing a question concerning physical reality in the first place. Insofar as the cognitive concept of questions and answers might be a crucial tool to any understanding of physical reality, a paramecium is, by construction, lacking the tools needed to understand physical reality. It presumably does not even understand what “understanding reality” means, in the sense that we mean the term. Ultimately, this is due to limitations in the very kind of cognitive capabilities of paramecia. And as just argued, we, too, almost surely have limitations in the very kind of our cognitive capabilities. So, the next (ironically self-referential) question in this essay:

11. Are there cognitive constructs of some sort, as fundamental as the very idea of questions and answers, that are necessary for understanding physical reality, and that are forever beyond our ability to even imagine due to the limitations of our brains, just as the notion of a question is forever beyond a paramecium?

It may help to clarify this question by emphasizing what it is not.

This question does not concern limitations on what we can know about what it is that we can never *know*.<sup>10</sup> Many things can be conceived by us humans even if they can never be known by us. The set of what it is that we cannot even *conceive of* is a (strictly smaller) subset of what it is that we cannot know. The issue I am concerned with is what we can ever perceive concerning that smaller set, the set of all that we cannot conceive of.

For example, I am not concerned here with the unknowables of how other branches of the many worlds of quantum mechanics turn out [48]. Nor am I concerned with values of variables that are unknown to us simply because we cannot directly observe them, e.g., values of variables concerning events outside of our Hubble sphere, or events within the event horizon of a black hole, or events like how many cells were in a dinosaur named “Bob” who lived in 90 million BC right where my house is now. The first one is calculable, in theory at least, even if not observable, and so is arguably knowable. The remaining three are not reflections of limits of our cognitive abilities — we can certainly conceive of those variables and their values without any difficulty. Rather, they can never be known by us for the simple reason that our ancillary engineering capabilities are not up

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<sup>10</sup>I am using the term “know” loosely here, since the distinctions between its various possible formal definitions [49] are not relevant to my arguments.

to the task, not for any reasons intrinsic to limitations of the science and math our minds can construct. They can be known, but we cannot find a path to such knowledge.

Also, I am not concerned here with the limitations of what can be known by humans that arise from the logical nature of self-referentiality, nor the limitations of Tarski [90] or Rice [64] or Chaitin [16, 29]. Indeed, the simple fact that we can prove that we cannot have this kind of knowledge means that we can conceive of having it. The impossibility theorems of Gödel and the computational hierarchies discussed shortly after question (6) are not the issues I wish to highlight at this point — after all, I am able to describe those kinds of unknowability. The concern here is what kinds of unknowable cognitive constructs might exist that we can never even be aware of, never mind describe (and never mind implement!).

In addition, I am not concerned with what cannot be known *with complete assurance*, for essentially statistical reasons. For example, I am not here concerned with the fact we can never be 100%, cross-my-heart-and-hope-to-die sure that a proof we write down has no logical errors, no matter how many times other people and computer programs might check it. It is conceivable that this limitation of what is knowable would apply no matter what kind of intelligence we had — whether we were beings far advanced evolutionarily from *Homo sapiens* or were just lowly paramecia.

Similarly, I am not here concerned with the theoretical impossibility of inductive inference. I am not concerned with the fact that if one restricts attention to future experiments, then, as Hume intuited [65], there is no assumption-free way to establish the validity of the scientific method [106, 105]. I am not concerned with the fact that none of machine learning can be formally justified in a non-trivial sense without making assumptions [100, 99, 105], that black-box optimization algorithms like simulated annealing cannot be justified without making assumptions [101], or that the use of Monte Carlo techniques to estimate expected values of functions cannot be justified without making assumptions [107].

Nor am I concerned with the kinds of impossibility results that would apply to any physical universe, regardless of its laws of physics, so long as entities within that physical universe could both encode questions concerning the state of that universe and answer them [103, 102, 104]. Like the paradoxes of Turing computability theory and mathematical logic mentioned above, these impossibilities lie within the remit of current (meta)mathematics rather than outside of it.

I am concerned with — and I wish to draw attention to — the issue of whether there are cognitive constructs that we cannot conceive of but that are as crucial to understanding physical reality as is the simple construct of a question. The paramecium cannot even conceive of the cognitive construct of a “question” in the first place, never mind formulate or answer a question; are there, similarly, cognitive constructs that we cannot conceive of, but that are just as necessary to

knowing all of physical reality as is the simple idea of questions and answers? I am emphasizing the possibility of things that are knowable, but not to us, because we are not capable of conceiving of that kind of knowledge in the first place.

This returns us to an issue that was briefly discussed above, in question (4), of how the set of what-we-can-conceive might evolve in the future. Suppose that, indeed, the kinds of knowledge concerning mathematical and physical reality that our current brains can conceive of having is a proper subset of all of the kinds of knowledge there are. In other words, suppose the set of what-can-be-known-but-not-even-conceived-by-us is non-empty. This leads to a new question, the final one of this essay:

12. Is there any way that we imagine testing — or at least gaining insight into — whether our SAM can, in the future, capture all of physical reality? If not, is there any way of gaining insight into how much of reality is forever beyond our ability to even conceive of? In short, what can we ever know about the nature of that which we cannot conceive of?

From a certain perspective, this question might appear to be a scientific version of a conspiracy theory, writ large. One might argue that it is no different in kind from the fact that: you can never prove that ghosts don't exist, either theoretically or empirically; that Marduk doesn't really pull the strings in human affairs, nor does an Abrahamic deity; that it is not actually the believers in QAnon who are members of a secret cabal of Satan-worshiping cannibalistic pedophiles running a global child sex-trafficking ring and engaged in a massive (highly successful) disinformation campaign.

Adopting this perspective, one might say, "Sure, yes, it may be that there are unconceivable aspects to reality. We can't somehow prove that there aren't. But by definition, we could never know such unconceivables — could never investigate them empirically — so who cares?" Isn't the ending question of this essay about can-never-be-conceived aspects of reality just a pompously inflated version of the platitude that you can't prove a negative? The famous scalpel of Occam suggests you should expunge consideration of such a proposition from your intellectual to-do list. Just as the scalpel denigrates the proposition that there is an Abrahamic deity in the sense that is defended by Deists, deriding that proposition as vacuous, said scalpel also suggests that the final question of this essay is vacuous, with no consequences (by definition), and therefore unworthy of contemplation.

There are at least three reasons to suspect that we actually *can* find the answer to (some aspects of) the question I am posing, which salvages the question from the realm of such vacuities:

- i) Most prosaically, we may be able to make some inroads into what is currently beyond our ability to conceive of we can ever somehow construct a

super-Turing computer [4] and exploit it to consider the question of what knowledge can never be conceived by us, thereby breaking free of the strictures of the Church — Turing thesis. More speculatively, as our cognitive abilities grow, we might at least be able to establish the existence of what we can never conceive, through some observational / simulational / theoretical / who knows? process. In other words, it may be that the feedback loop between our extended minds and our technology does let us break free of the evolutionary accident — of the fact that our minds were formed for the Veldt — at least in certain regards. It may be that the answer to question 4 above is “no” — and this may have consequences.

- ii) Suppose we encounter extraterrestrial intelligence, e.g., by plugging into some vast galaxy-wide web of interspecies discourse, containing in particular some cosmic version of Stack Exchange. To answer the question — to determine whether there are aspects of physical reality that are knowable but that we humans cannot even conceive of that kind of knowledge — might require nothing more than our posing that question to the cosmic Stack Exchange, and then reading through the answers that get posted.
- iii) Consider our evolutionary progeny — not just future variants of our species that evolve from us via conventional Neo-Darwinian mutation, crossover, and selection, but future members of any species (organic and/or inorganic) that we consciously design. It seems quite likely that the minds of such successors of ours will have a larger set of things that can be conceived than we do. After all, the decades-long experiments conducted in Richard Lenski’s lab [20] show that even without any consciously directed intervention (relying solely on the proverbial Neo-Darwinian “blind watchmaker” to guide the evolution, albeit a sped-up watchmaker), there is unceasing growth in adaptive fitness. There is no reason to believe that this would not also apply to cognitive “fitness.” (See the discussion above of a time-series analysis of the set of cognitive capabilities of terrestrial species.)

In addition, the successors that we design would be an evolutionary first: a species directly constructed in a goal-oriented, Lamarkian manner. This novelty in how our descendant species will be constructed may mean that their cognitive “genotype” will have jumped out of our current metastable state [5, 72]. The jump from us to them may be a massive punctuated equilibrium, overcoming the limitations imposed on us by the frozen accidents in our evolutionary past.

It seems likely that these children of ours, with their attendant cognitive advances beyond us, will be here within 50 — 100 years. Presumably we will go extinct soon after their arrival (like all good parents making way for

their children). So, on our way out the door, as we gaze up at our successors in open-mouthed wonder, as one of our last acts:

We can simply ask our question of them.

## Acknowledgments:

I would like to thank David Kinney for interesting conversation on these issues. This work was supported by funding from the Santa Fe Institute.

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